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Preprojective Partitions for Selfinjective Algebras

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1. INTRODUCTION

In this paper we investigate upper bounds for the length of the preprojective partition for selfinjective algebras. The preprojective partition has been defined for arbitrary Artin algebras by Auslander and Smalø in [2]. For a representation-finite algebra, that is, an algebra Λ for which the set $\operatorname{ind}\Lambda$ of isomorphism classes of finitely generated indecomposable Λ-modules is finite, the preprojective partition can be defined as the unique decomposition of $\operatorname{ind}\Lambda$ into disjoint non-empty sets $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{p(\Lambda)}$ obtained inductively as follows: $\mathbf{P}_i$ is the smallest subset of $\mathbf{Q}_i = \operatorname{ind}\Lambda \backslash (\mathbf{P}_0 \cup \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_{i-1})$ such that for every module M in an isomorphism class in $\mathbf{Q}_i$ there is an epimorphism from a finite direct sum of representatives of isomorphism classes in $\mathbf{P}_i$ onto M. The preinjective partition $\operatorname{ind}\Lambda = \mathbf{I}_0 \cup \mathbf{I}_1 \cup \cdots \cup \mathbf{I}_{i(\Lambda)}$ with $\mathbf{I}_k \neq \varnothing$ is defined dually.

D. Zacharia proved that $p(\Lambda)$ and $i(\Lambda)$ coincide if Λ is a hereditary representation-finite algebra and that $p(\Lambda)+1$ equals the maximum of the lengths of the indecomposable Λ-modules. If the graph underlying the quiver of Λ is the Dynkin-diagram A_n, D_n, E_6, E_7, or E_8, then $p(\Lambda)$ equals $n-1$, $2n-4$, 10, 16, or 28, respectively [9].

We will prove the following theorem:

THEOREM. *Let Λ be a finite-dimensional selfinjective algebra over an algebraically closed field k and assume Λ is representation-finite. If Λ is of tree-class T_Λ, then $p(\Lambda) \leqslant l(T_\Lambda)$ and $i(\Lambda) \leqslant l(T_\Lambda)$, where $l(T)$ equals n, $2n-3$, 11, 17, or 29 for $T = A_n$, D_n, E_6, E_7, or E_8, respectively.*

The tree-class T_Λ is defined as follows: The stable part ${}_s\Gamma_\Lambda$ of the Auslander–Reiten quiver Γ_Λ of Λ has the form $\mathbb{Z}T/G$, where $T = T_\Lambda$ is a Dynkin-diagram A_n, D_n, E_6, E_7, or E_8 and G is an admissible group of automorphisms of $\mathbb{Z}T$ [5, 6].

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We will give an example of a selfinjective algebra Λ for which $p(\Lambda) \neq i(\Lambda)$, in contrast to the hereditary case. In this example, both numbers are less than $l(T_\Lambda)$. Another example Λ' will show that $p(\Lambda')$ and $i(\Lambda')$ may coincide and still be less than $l(T_{\Lambda'})$.

However, it is not possible to find a better bound for $p(\Lambda)$ or $i(\Lambda)$ which holds for all selfinjective algebras of a given tree-class T. More precisely, for every Dynkin-diagram $T = A_n$, D_n, E_6, E_7, or E_8, there is a selfinjective algebra Λ of tree-class T for which $p(\Lambda) = i(\Lambda) = l(T)$. It can be obtained as follows. Choose a hereditary algebra A over an algebraically closed field k whose quiver has T as its underlying graph and let Λ be the trivial extension of A by the A–A-bimodule $\mathrm{Hom}_k(A, k)$. The results of Tachikawa [8] imply that Λ is weakly symmetric (and hence selfinjective) of tree-class T, and Rohnes has shown that $p(\Lambda) = i(\Lambda) = l(T)$ [7].

2. Proof of the Theorem

Let Λ be a representation-finite Artin algebra. We call the composition $g_m \cdots g_1$, $m \geqslant 1$, of irreducible morphisms

$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} X_m$$

a chain (of irreducible morphisms) of length m, and we say that such a chain is non-zero if $g_m \cdots g_1 \neq 0$. If X and Y are indecomposable Λ-modules, every non-isomorphism $f: X \to Y$ can be written as a sum of chains of irreducible morphisms [1]. Moreover, there is a bound for the length of non-zero chains, since any chain from an indecomposable X to itself lies in the radical of End X, which is nilpotent. By $m(\Lambda)$ we denote the maximal possible length of a non-zero chain of irreducible morphisms from an indecomposable projective module to its simple top.

Proposition. *For a representation-finite Artin algebra Λ the inequality $p(\Lambda) \leqslant m(\Lambda)$ holds.*

Proof. For the sake of simplicity we say that an indecomposable module X lies in $\mathbf{P}_i$ if its isomorphism class does. By the definition of the preprojective partition we can find an epimorphism from a finite direct sum of modules in $\mathbf{P}_{i-1}$ onto X_i for any X_i in $\mathbf{P}_i$. Let $p = p(\Lambda)$ and choose in $\mathbf{P}_p$ a module X_p, which is simple by [3]. Starting with X_p and using induction on $p - i$, we find an indecomposable X_i in $\mathbf{P}_i$ and a morphism $f_{i+1}: X_i \to X_{i+1}$ such that $f_p \cdots f_{i+1} \neq 0$ for each $i = 0, \ldots, p-1$. Note that X_0 is an indecomposable projective and hence is the projective cover of X_p. Each morphism f_i can be written as a sum $\sum_j g_{ij}$ of chains of irreducible morphisms, as noted above. Because $f_p \cdots f_1 \neq 0$, we find a chain $g_{ij(i)}: X_{i-1} \to X_i$ for each i such that $g_{pj(p)} \cdots g_{1j(1)} \neq 0$. Thus, there is a non-zero chain from the projective indecomposable X_0 to its top X_p whose length is at least p, and we conclude $p = p(\Lambda) \leqslant m(\Lambda)$.

PROPOSITION. *Let Λ be a finite-dimensional selfinjective algebra over an algebraically closed field k and assume Λ is representation-finite. Then $m(\Lambda)$ equals $l(T_\Lambda)$.*

This proposition together with the previous one yields the bound for $p(\Lambda)$ claimed in the theorem. The bound for $i(\Lambda)$ follows from the fact that the opposite algebra Λ^{op} is selfinjective of the same tree-class and that $i(\Lambda) = p(\Lambda^{\mathrm{op}})$.

Proof of Proposition. Denote by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules, and let the stable category $\underline{\operatorname{mod}}\,\Lambda$ be the category obtained from $\operatorname{mod}\Lambda$ as the residue category modulo the ideal of morphisms factoring through a projective module. In the first step we show that $m(\Lambda) - 1$ is the maximum of the lengths of non-zero chains of irreducible morphisms stopping at a simple Λ-module in $\underline{\operatorname{mod}}\,\Lambda$. Working in $\underline{\operatorname{mod}}\,\Lambda$ we then prove $m(\Lambda) = l(T_\Lambda)$.

Every non-zero chain from an indecomposable X to a simple S in $\operatorname{mod}\Lambda$ can be extended to a non-zero chain from the projective cover P of S to S. On the other hand, composing any non-zero chain from P to S with the inclusion of the radical of P into P, which is the only irreducible morphism stopping at P up to isomorphism, yields the zero morphism. We infer that $m(\Lambda)$ is the maximal length of a non-zero chain ending at a simple. The projective cover P of S is the only projective from which there is a non-zero chain $f_s \cdots f_1$ to S, and the first morphism f_1 in such a chain is isomorphic to the canonical projection $P \to P/\operatorname{soc} P$, since P is injective too. So $m(\Lambda) - 1$ equals the maximal length of a non-zero chain $g_r \cdots g_1$ from a non-projective indecomposable X to a simple S. Because $g_r \cdots g_1$ is an epimorphism, it does not factor through a projective. Consequently, $m(\Lambda) - 1$ is the maximal length of a non-zero chain in $\underline{\operatorname{mod}}\,\Lambda$ stopping at a simple Λ-module.

Let $k(\mathbb{Z}T)$ be the mesh-category associated with the translation-quiver $\mathbb{Z}T = \mathbb{Z}T_\Lambda$ and let $F: k(\mathbb{Z}T) \to \underline{\operatorname{mod}}\,\Lambda$ be a well-behaved functor [5, 6]. Then F induces a bijection

$$\coprod_{Fz = Fy} \operatorname{Hom}_{k(\mathbb{Z}T)}(z, x) \to \underline{\operatorname{Hom}}_\Lambda(Fy, Fx)$$

for any pair x, y of vertices of $\mathbb{Z}T$, where $\underline{\operatorname{Hom}}_\Lambda$ denotes the vector space of morphisms in $\underline{\operatorname{mod}}\,\Lambda$. Let x be a fixed vertex in $\mathbb{Z}T$. Choose a path $w: z \to x$ in $\mathbb{Z}T$ of length r and denote the residue class of w in $k(\mathbb{Z}T)$ by $\bar{w}$. Then $F\bar{w}$ is a chain of irreducible morphisms from Fz to Fx of length r, and $F\bar{w}$ is non-zero if $\bar{w} \neq 0$. On the other hand, any chain of irreducible morphisms $g_r \cdots g_1$ in $\underline{\text{mod}}\ \Lambda$ stopping at Fx can be written as $\sum F\bar{w}$, where w ranges over paths stopping at x whose length is not less than r. We infer that for a path $w: z \to x$ with $\bar{w} \neq 0$, having maximal length among all paths $v: t \to x$ with $\bar{v} \neq 0$, the image $F\bar{w}$ is a non-zero chain of irreducible morphisms in $\underline{\text{mod}}\ \Lambda$ stopping at Fx of maximal length. In particular, $m(\Lambda) - 1$ equals the maximal length of a path w stopping at a vertex x with Fx simple and such that $\bar{w} \neq 0$.

Let x be an arbitrary vertex of $\mathbb{Z}T$. In [4] it was shown that $\text{Hom}_{k(\mathbb{Z}T)}(\nu^{-1}x, x) = k$, where ν is the Nakayama-permutation of $\mathbb{Z}T$, and moreover that any vertex y with $\text{Hom}_{k(\mathbb{Z}T)}(y, x) \neq 0$ lies between the slice starting at $\nu^{-1}x$ and the one stopping at x. So there is a path $w: \nu^{-1}x \to x$ with $\bar{w} \neq 0$, and no path $v: y \to x$ with $\bar{v} \neq 0$ is longer than w. Using the formulas for ν given in [4] it is easy to see that the length of w equals $l(T) - 1$. (Notice that in the case of E_6 the formula should be $\nu(p, q) = (p + q + 2, 6 - q)$ for $q \leqslant 5$.) We conclude that $m(\Lambda) - 1 = l(T) - 1$.

3. Examples

We now turn to the two examples described in the introduction. In our first example, Λ is given by the quiver

$$\begin{tikzcd} & 1 \arrow[dl, shift right, "\alpha_1"'] & \\ 2 \arrow[ur, shift right, "\alpha_2"'] \arrow[dr, "\beta_2"'] & & 0 \arrow[ll, "\beta_0"] \arrow[loop right, "\alpha_0"] \\ & 3 \arrow[ur, "\beta_3"'] & \end{tikzcd}$$

and the relations $\beta_0 \alpha_0 = \alpha_2 \beta_0 = \beta_2 \alpha_1 = \alpha_0 \beta_3 = 0$ and $\alpha_0^3 = \beta_3 \beta_2 \beta_0$, $\alpha_1 \alpha_2 = \beta_0 \beta_3 \beta_2$. We display a representation

$$\begin{tikzcd} & V(1) \arrow[dl, shift right, "V(\alpha_1)"'] & \\ V(2) \arrow[ur, shift right, "V(\alpha_2)"'] \arrow[dr, "V(\beta_2)"'] & & V(0) \arrow[ll, "V(\beta_0)"] \arrow[loop right, "V(\alpha_0)"] \\ & V(3) \arrow[ur, "V(\beta_3)"'] & \end{tikzcd}$$

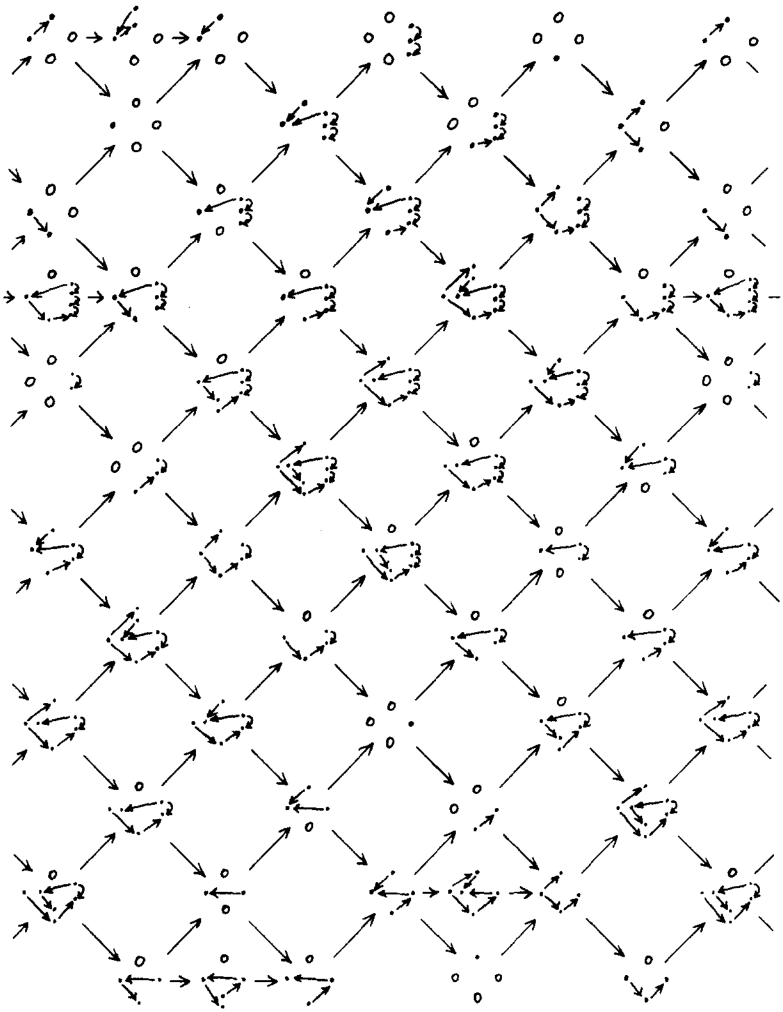

FIGURE 1

of our quiver by dots and arrows: Each dot at a vertex i corresponds to a basis vector of $V(i)$; given an arrow $\gamma: i \to j$ in the quiver, an arrow is drawn from a dot at i to a dot at j if $V(\gamma)$ maps the corresponding basis vector of $V(i)$ onto the one of $V(j)$, and $V(\gamma)$ annihilates the remaining basis vectors of $V(i)$. Figure 1 shows a part of the universal cover of the Auslander–Reiten quiver Γ_Λ of Λ [6], containing each indecomposable Λ-module at least once.

Clearly, Λ is of tree-class A_{12}, so that $l(T_\Lambda)=12$. In Fig. 2 we draw the same part of the universal cover of Γ_Λ, but we replace each indecomposable X by a pair of integers (i, j), indicating that X lies in $\mathbf{P}_i$ and in $\mathbf{I}_j$. Inspection shows that $p(\Lambda)=8$ and $i(\Lambda)=9$.

Our second example Λ' is defined by the quiver

$$1 \underset{\beta_1}{\overset{\beta_0}{\leftrightarrows}} 0 \circlearrowleft \alpha_0$$

and the relations $\beta_0 \alpha_0 = \alpha_0 \beta_1 = 0$ and $\alpha_0^3 = \beta_1 \beta_0$. For the Λ'-modules we use

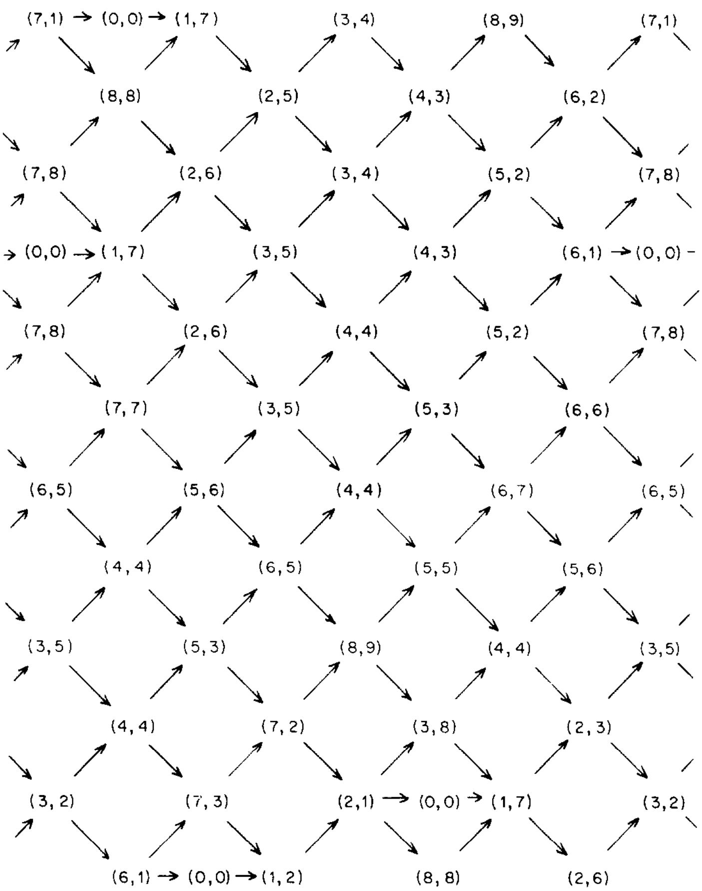

FIGURE 2

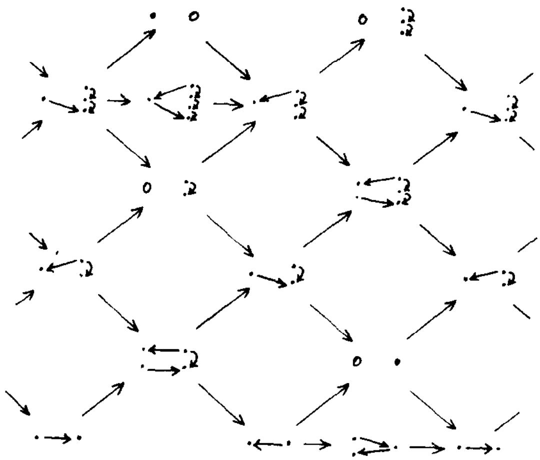

FIGURE 3

the notation introduced above. Figure 3 shows a part of the universal cover of $\Gamma_{\Lambda'}$ from which one can see that Λ' is of tree-class A_6. We can read off that $p(\Lambda') = i(\Lambda') = 5$ from Fig. 4.

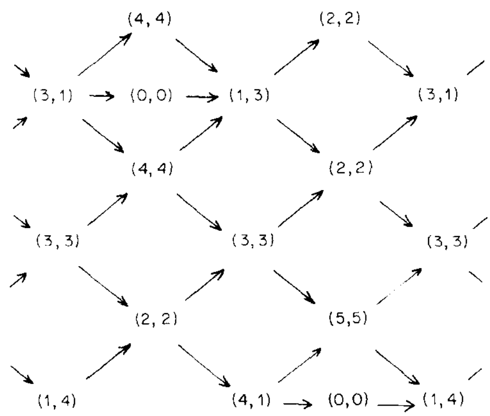

FIGURE 4

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